Distributed Control-Estimation Synthesis for Stochastic Multi-Agent Systems via Virtual Interaction Between Non-Neighboring Agents

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***Abstract*—This letter considers the optimal distributed control problem for a linear stochastic multi-agent system (MAS). Due to the distributed nature of MAS network, the information available to an individual agent is limited to its vicinity. From the entire MAS aspect, this imposes the structural constraint on the control law, making the optimal control law computationally intractable. This letter attempts to relax such a structural constraint by expanding the neighboring information for each agent to the entire MAS, enabled by the distributed estimation algorithm embedded in each agent. By exploiting the estimated information, each agent is not limited to interact with its neighborhood but further establishing the 'virtual interactions' with the non-neighboring agents. Then the optimal distributed MAS control problem is cast as a synthesized control-estimation problem. An iterative optimization procedure is developed to find the control-estimation law, minimizing the global objective cost of MAS.**

***Index Terms*—Distributed control, optimal control.**

I. Introduction

DISTRIBUTED control within a cooperative multi-agent system (MAS) is the key enabling technology for different networked dynamical systems [1]–[4]. Notwithstanding diverse distributed control strategies, their optimality is one of the stumbling blocks due to individual agents' limited information. In particular, finding the optimal distributed control with network topological constraint is a well-known NP-hard problem [5]. To ease this problem, some former studies have focused on a specific form of objective function along with certain MAS network topology conditions under which the optimal distributed control laws can be designed [6]. More particularly, different techniques have been investigated to design suboptimal distributed control laws for different MAS cooperative tasks [5], [7], [8]. In this letter, a new avenue for accomplishing the optimal distributed control of MAS is presented while not requiring the restricted form of the objective function, nor the network topology. The key idea is to expand the available information for each agent by employing the distributed estimation algorithm, and use the expanded information to relax the network topological constraint in a tractable manner. In a nutshell, the main contributions are the following.

1) A synthesized distributed control-estimation framework is proposed based on the authors' preliminarily developed distributed estimation algorithm [9]. The newly proposed framework enables the interactions between non-neighboring agents, namely *virtual interactions*.
2) With the aid of virtual interactions, a design procedure that solves for the optimal distributed control law of the stochastic MAS over a finite time horizon is developed, which was originally an intractable non-convex problem due to the network topological constraint.

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II. Problem Formulation

A. Dynamical Model of Stochastic MAS

Consider a stochastic linear multi-agent dynamical system including N homogeneous agents whose dynamics is given by:

$$x_i(t+1) = Ax_i(t) + Bu_i(t) + w_i(t), \quad \forall i \in \{1, \ldots, N\} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^p$ are the state and the control input of the i^{th} agent, respectively. $w_i(t)$ is a disturbance imposed on the i^{th} agent, assumed to follow zero-mean white Gaussian distribution with covariance $\Theta_i(t) \succ 0$. $t \in \mathbb{Z}_+ = \{0, 1, \ldots, \}$ indicates a discrete-time index. A, B are the system matrices with appropriate dimensions, and are assumed to satisfy the controllability condition. Accordingly, the entire MAS dynamics can be written as

$$\begin{aligned} x(t+1) &= \tilde{A}x(t) + \tilde{B}u(t) + \tilde{w}(t) \\ \tilde{A} &= (I_N \otimes A), \ \ \tilde{B} = (I_N \otimes B) \\ x(t) &= \left[x_1^{\mathrm{T}}(t) \cdots x_N^{\mathrm{T}}(t)\right]^{\mathrm{T}}, \ \ u(t) = \left[u_1^{\mathrm{T}}(t) \cdots u_N^{\mathrm{T}}(t)\right]^{\mathrm{T}} \\ \tilde{w} &= \left[w_1^{\mathrm{T}}(t) \cdots w_N^{\mathrm{T}}(t)\right]^{\mathrm{T}} \end{aligned} \tag{2}$$

where $\otimes$ is the Kronecker product between matrices. The interactions between agents are rendered by inter-agent network topology, described by a graph model $\mathcal{G}$ consisting of a node set $\mathcal{V} = \{1, 2.., N\}$ indexing each agent and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ indicating the network connectivity between the agents. Each edge $(i, j) \in \mathcal{E}$ denotes that the node i can acquire the state information of the node j. An adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ can express the network connectivity of the graph model, where its element $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. A degree matrix is defined as $\mathcal{D} = diag(d_1 \cdots d_N)$ where $d_i = \sum_j a_{ij}$ is (weighted) degree of node i. The Laplacian matrix $\mathcal{L}$, given by $\mathcal{L} = \mathcal{D} - \mathcal{A}$, is useful for analysis of the network topology. The set of agents whose state information is available to the i^{th} agent, i.e., the neighborhood of the i^{th} agent, is expressed as Ω_i, and its cardinality is expressed as $|\Omega_i|$. Based on the given network topology, the noisy measurement of neighborhood states $\{x_j(t)|j \in \mathcal{V}\}$ from the i^{th} agent's perspective can be represented as follows [9]:

$$z_{ij}(t) = c_{ij}\big(x_j(t) + v_{ij}(t)\big), \quad \forall j \in \mathcal{V} \tag{3}$$

where c_{ij} indicates the availability of the measurement of the j^{th} agent's state from the i^{th} agent such that $c_{ij} = 1$ when $j \in \Omega_i$, and $c_{ij} = 0$ otherwise. The noise of the measurement from the i^{th} to the j^{th} agent is specified as $v_{ij}(t)$ which is assumed to be independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and covariance $\Xi_{ij}(t) \succ 0$. Further the measurement and the noise sets of the i^{th} agent are denoted by $Z_i(t) = [z_{i1}^{\mathrm{T}}(t) \cdots z_{iN}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $v_i(t) = [v_{i1}^{\mathrm{T}}(t) \cdots v_{iN}^{\mathrm{T}}(t)]^{\mathrm{T}}$, respectively. Over a finite time horizon $t = 0, \ldots, T$, one can rewrite (2) as a static form by stacking up the variables and matrices [10]:

$$x = P_{11}w + P_{12}u \tag{4}$$

where

$$P_{11} = (I - \mathrm{D}\bar{A})^{-1},\ P_{12} = (I - \mathrm{D}\bar{A})^{-1}\mathrm{D}\bar{B}$$
$$\bar{A} = I_{T+1} \otimes \tilde{A},\ \bar{B} = \begin{bmatrix} I_T \otimes \tilde{B} \\ 0_{Nn\times NpT} \end{bmatrix}$$
$$\mathrm{D} = \begin{bmatrix} 0_{Nn\times NnT} & 0_{Nn\times Nn} \\ I_{NnT} & 0_{NnT\times Nn} \end{bmatrix}$$

where I_T and 0_T respectively denote the identity and zero matrices of size $T \times T$, and $M_i = [0_p \cdots I_p \cdots 0_p] \in \mathbb{R}^{p\times Np}$ is the block matrix having I_p in the i^{th} block entry and filled with 0_p in other block entries. And $x = [x(0)^{\mathrm{T}} \cdots x(T)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn(T+1)}$, and $u = \sum_i^N (I_T \otimes M_i^{\mathrm{T}})u_i \in \mathbb{R}^{NpT}$ are the stacked agents' states and their control inputs over the horizon, where $u_i = [u_i(0)^{\mathrm{T}} \ldots u_i(T-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{pT}, \forall i \in \mathcal{V}$. $w = [x(0)^{\mathrm{T}}\ \tilde{w}(0)^{\mathrm{T}}\ \ldots \tilde{w}(T-1)^{\mathrm{T}}] \in \mathbb{R}^{Nn(T+1)}$ is the vector containing initial agents' states and the additive noise. Over the finite time horizon T, individual agents interact with their neighbors according to the control law u_i embedded in each agent. Without loss of generality, u_i can be designed by the following output feedback control law [10]:

$$u_i = (I_T \otimes M_i)\mathcal{F}Z_{i,(0:T-1)} = (I_T \otimes M_i)\mathcal{F}C(x + v_i),\ \forall i \in \mathcal{V} \tag{5}$$

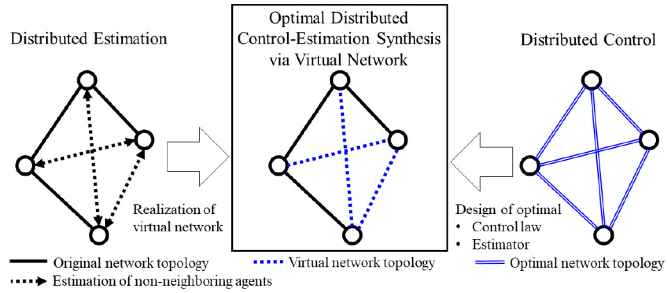

Fig. 1. Virtual interaction based distributed control-estimation synthesis.

where $v_i = [v_i(0)^{\mathrm{T}} \ldots v_i(T)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn(T+1)}, \forall i \in \mathcal{V}$, $Z_{i,(0:T-1)} = [Z_i(0)^{\mathrm{T}} \cdots Z_i(T-1)^{\mathrm{T}}] \in \mathbb{R}^{NnT}$, and $C = [I_{NnT}\ 0_{NnT\times Nn}]$. The crucial part is the design of the feedback gain, which is denoted by $\mathcal{F} \in \mathbb{F}$. Here, $\mathbb{F} \subset \mathbb{R}^{NpT\times NnT}$ is an invariant subspace that encodes network topological constraints for distributed MAS imposed by $\mathcal{A}$, as well as embeds causal feedback policies by forcing the future response entries to zeros.

B. Optimal MAS Distributed Control Problem

Given the equivalent static form of the stochastic linear MAS dynamics over the time horizon T (4), we seek to address the optimal distributed control problem.

Problem 1 (Optimal Distributed Control Law Subject to Structural Constraint) [10]:

$$\min_{\mathcal{F}\in\mathbb{F}}\ \mathbb{E}\big[x^{\mathrm{T}}\mathcal{Q}x + u^{\mathrm{T}}\mathcal{R}u\big]$$
$$\text{subject to}\ \ (4),\ (5),\ \forall i \in \mathcal{V} \tag{6}$$

where $\mathcal{Q} \in \mathbb{R}^{Nn(T+1)\times Nn(T+1)} \succeq 0$, and $\mathcal{R} \in \mathbb{R}^{NpT\times NpT} \succ 0$ are the associated weight matrices.

Due to the structural constraints imposed on the control input space $\mathbb{F}$, Problem 1 is a highly non-convex problem, which is indeed NP-hard and a formidable computational burden [5]. To circumvent such a difficulty, we propose a concept of *virtual network topology* that allows for the interactions between non-neighboring agents, i.e., *virtual interaction* as depicted in Fig. 1.

Since the state information of the non-neighboring agent is not available, an appropriate estimator is required for each agent to obtain the estimates of non-neighboring agents' states. Using the Bayesian approach, Kalman-like filter[1] is adopted for estimation as we consider a linear MAS along with the Gaussian uncertainties.

Definition 1: The state estimate and its covariance of the MAS using the i^{th} agent's measurements are denoted by ${}^i\hat{x}(t) := \mathbb{E}[x(t)|Z_{i,(0:t)}]$ and ${}^i\Sigma(t) := \mathbb{E}[(x(t) - {}^i\hat{x}(t))(x(t) - {}^i\hat{x}(t))^{\mathrm{T}}|Z_{i,(0:t)}], \forall j \in \mathcal{V}$, respectively [9], where $\mathbb{E}[\bullet|\bullet]$ is the conditional expectation.

The nominal recursive structure of Kalman-like filter is represented by:

$${}^i\hat{x}(t) = {}^i\hat{x}^-(t) + L_i(t)H_i\big(Z_i(t) - {}^i\hat{x}^-(t)\big) \tag{7}$$

[1]We refer to the recursive linear estimation algorithm that estimates the states via optimizing error covariances as the Kalman-like filter.

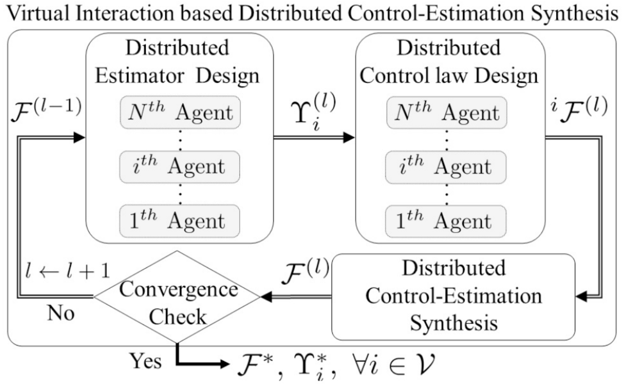

Fig. 2. Iterative optimization procedure for the optimal distributed control-estimation synthesis.

where ${}^{i}\hat{x}^{-}(t) := \mathbb{E}[x(t)|Z_{i,(0:t-1)}]$ denotes the predicted state estimate from the i^{th} agent's perspective. $H_i \in \mathrm{R}^{n|\Omega_i|\times nN}$ only encodes the neighbor of the i^{th} agent, that is, $H_i = [h_1 \ \ h_2 \ \ \cdots h_{|\Omega_i|}]^{\mathrm{T}} \otimes I_n$ where $h_m \in \mathbb{R}^N$, and $m = 1, 2, \dots, |\Omega_i|$ are the nonzero column vectors of the matrix $diag(c_{i1}, c_{i2}, \dots, c_{iN})$. And, $L_i(t) \in \mathbb{R}^{nN\times n|\Omega_i|}$ represents the estimator gain at time step t for estimating the states of the MAS from the i^{th} agent's perspective. Once the entire MAS state estimate information becomes available for each agent, one can replace (5) with the estimation-based feedback control law. Accordingly, Problem 1, distributed control law subject to structural constraint, can be reformulated into the problem that simultaneously designs both distributed control and distributed estimator.

Problem 2 (Optimal Distributed Control-Estimation law With Virtual Interactions):

$$\begin{aligned} &\min_{\mathcal{F}\in\tilde{\mathbb{F}}, \Upsilon_i, \forall i\in\mathcal{V}} \quad J(\mathcal{F}, \Upsilon_1, \dots, \Upsilon_N) \\ &\quad \text{subject to (1), and} \\ &\qquad\qquad u_i = (I_T \otimes M_i)\mathcal{F}C\ {}^{i}\hat{x},\ \ \forall i \in \mathcal{V} \qquad (8) \end{aligned}$$

where ${}^{i}\hat{x} = [{}^{i}\hat{x}(0)^{\mathrm{T}} \cdots {}^{i}\hat{x}(T)^{\mathrm{T}}]^{\mathrm{T}}$, $J(\mathcal{F}, \Upsilon_1, \dots, \Upsilon_N) := \mathbb{E}[x^{\mathrm{T}}\mathcal{Q}x + u^{\mathrm{T}}\mathcal{R}u]$ and $\Upsilon_i := \{L_i(t)|t = 0, \dots, T\}$ is the set of estimator gains over the time horizon T for the i^{th} agent.

Remark 1: It is worth noting that, compared to $\mathbb{F}$, $\tilde{\mathbb{F}} \subset \mathbb{R}^{NpT\times NnT}$ is a subspace that only encodes causal feedback policies, not restricted by any network topological constraint.

Albeit Problem 2 can successfully relax the structural constraint on the control law $\mathcal{F}$, it is not straightforward to solve as the control law and the state estimation errors mutually affect each other [9]. To resolve such complexity, we propose an iterative optimization procedure in a distributed fashion such that: i) divide the primal problem (Problem 2) into the set of sub-problems, each is viewed from an individual agent's perspective; ii) sequentially design the distributed estimation and control laws for each sub-problem; iii) mix the results from individual sub-problems to approximate the optimal solution to the primal problem. The overall schematic of the proposed distributed control-estimation synthesis is delineated in Fig. 2.

For the l^{th} iteration, the optimization procedure consists of the following sub-steps. Firstly, *distributed estimator design* optimizes a set of estimator gains $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$ based on the disturbance/noise model, the network topological constraint, and the suboptimal control law resulted from the previous iteration. Then, *distributed control law design* computes a set of optimal control laws ${}^{i}\mathcal{F}^{(l)}, \forall i \in \mathcal{V}$, each from the individual agents' perspectives, based on the state estimation error information from the designed distributed estimator. Finally, *distributed control-estimation synthesis* mixes the set of ${}^{i}\mathcal{F}^{(l)}, \forall i \in \mathcal{V}$ to construct the solution candidate, $\mathcal{F}^{(l)}$, for Problem 2. The constructed control law is evaluated to check the convergence and is used for the next iteration. The iteration terminates once the pre-defined stopping criteria are fulfilled, yielding the optimal control-estimation law, denoted by $\mathcal{F}^*$ and $\Upsilon_i^*, \forall i \in \mathcal{V}$.

III. Algorithm Development

This section details out the proposed synthesis procedure of the optimal distributed control-estimation law that can comply with an arbitrary network topology of the stochastic MAS.

A. Distributed Estimator Design

To begin with, the distributed estimation algorithm is optimized by means of estimator gains $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$. As offline design phase, individual estimators can be designed based on the entire MAS model information along with the control law computed from the previous iteration, (A, B, and $\mathcal{F}^{(l-1)}$). For brevity, we use $\mathcal{F}$ to designate $\mathcal{F}^{(l-1)}$ in this subsection. Recalling (7), the distributed estimator embedded in the i^{th} agent calculates ${}^{i}\hat{x}(t), \forall t \in \{0, \dots, T\}$, whose performance can be measured by the estimation error.

Definition 2: ${}^{i}e(t) := x(t) - {}^{i}\hat{x}(t)$ is the MAS state estimation error from the i^{th} agent's perspective, and its covariance is ${}^{i}\Sigma(t)$ by Definition 1. Further, $e(t) = [{}^{1}e(t)^{\mathrm{T}} \cdots {}^{N}e(t)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{NnN}$ stacks all the estimation errors from individual agents' estimator, and the corresponding covariance is denoted by $\Sigma(t) := \mathbb{E}[e(t)e(t)^{\mathrm{T}}] \in \mathbb{R}^{NnN\times NnN}$. Similarly, ${}^{i}e^{-}(t)$, ${}^{i}\Sigma^{-}(t)$, $e^{-}(t)$, $\Sigma^{-}(t)$ are defined in terms of the predicted state estimate ${}^{i}\hat{x}^{-}(t)$ [9].

Assumption 1: The initial conditions ${}^{i}\hat{x}(0), \forall i \in \mathcal{V}$, and $\Sigma(0)$ are given to individual agents in order to initiate each of their distributed estimators.

Based on the prior knowledge, the estimation based control input of the i^{th} agent at time step t can be written by:

$$u_i(t) = M_i \sum_{k=0}^{t} \mathcal{F}_{kt}\ {}^{i}\hat{x}(k) \tag{9}$$

where $\mathcal{F}_{kt} \in \mathbb{R}^{pN\times nN}$ is the block matrix which spans from $(knN)^{th}$ to $(knN + nN - 1)^{th}$ columns and from $(kpN)^{th}$ to $(kpN + pN - 1)^{th}$ rows of the control law matrix $\mathcal{F}$. With (9), the entire MAS dynamics (2) can be expressed by:

$$\begin{aligned} x(t+1) = \tilde{A}x(t) + \tilde{B}\mathcal{F}_{tt}x(t) + \sum_{k=0}^{t-1}\tilde{B}\mathcal{F}_{kt}x(k) \\ - \sum_{k=0}^{t}\bar{B}\tilde{M}\tilde{\mathcal{F}}_{kt}e(k) + \tilde{w}(t) \end{aligned} \tag{10}$$

where $\bar{B} = 1_N^{\mathrm{T}} \otimes \tilde{B}$, $\tilde{M} = blkdg(M_1^{\mathrm{T}}M_1, \dots, M_N^{\mathrm{T}}M_N) \in \mathbb{R}^{NpN\times NpN}$, and $\tilde{\mathcal{F}}_{kt} = I_N \otimes \mathcal{F}_{kt}$. $blkdg(\bullet)$ denotes a block-diagonal matrix with block matrices $\bullet$, and the vector $1_N \in \mathbb{R}^N$ indicates every element equals to 1. Then the predicted state estimate of the entire MAS from the i^{th} agent's perspective is given by:

$$ {}^i\hat{x}^-(t+1) = \tilde{A}\ {}^i\hat{x}(t) + \tilde{B}\mathcal{F}_{tt}{}^i\hat{x}(t) + \sum_{k=0}^{t-1} \tilde{B}\mathcal{F}_{kt}{}^i\hat{x}(k) \qquad (11) $$

Subtracting (11) from (10), and concatenating the results for all agents in $\mathcal{V}$ gives:

$$ e^-(t+1) = \Lambda_t e(t) + \sum_{k=0}^{t-1} \Psi_{kt} e(k) + 1_N \otimes \tilde{w}(t) $$
$$ \text{where } \Lambda_t = I_N \otimes (\tilde{A} + \tilde{B}\mathcal{F}_{tt}) - 1_N \otimes \bar{B}\tilde{M}\tilde{\mathcal{F}}_{tt}, $$
$$ \Psi_{kt} = I_N \otimes \tilde{B}\mathcal{F}_{kt} - 1_N \otimes \bar{B}\tilde{M}\tilde{\mathcal{F}}_{kt} \qquad (12) $$

Correspondingly, $\Sigma^-(t+1)$ is represented by:

$$ \begin{aligned} \Sigma^-(t+1) &= \Lambda_t \Sigma(t) \Lambda_t^{\mathrm{T}} + \Sigma_{\tilde{w}}(t) \\ &+ \sum_{q=0}^{t-1} \Lambda_t \mathbb{E}[e(t)e(q)^{\mathrm{T}}]\Psi_{qt}^{\mathrm{T}} + \sum_{p=0}^{t-1} \Psi_{pt}\mathbb{E}[e(p)e(t)^{\mathrm{T}}]\Lambda_t^{\mathrm{T}} \\ &+ \sum_{p=0}^{t-1}\sum_{q=0}^{t-1} \Psi_{pt}\mathbb{E}[e(p)e(q)^{\mathrm{T}}]\Psi_{qt}^{\mathrm{T}} \end{aligned} \qquad (13) $$

where $\Sigma_{\tilde{w}}(t) = (1_N 1_N^{\mathrm{T}}) \otimes blkdg(\Theta_1(t), \dots, \Theta_N(t))$. Note that the summation terms in the RHS of (13) (e.g., $\mathbb{E}[e(p)e(q)^{\mathrm{T}}], q \neq p$) imply the correlations of state estimates over time induced by the control law (9). Now, the predicted error, (7) can be rewritten by:

$$ \begin{aligned} {}^i\hat{x}(t+1) &= {}^i\hat{x}^-(t+1) \\ &\quad + L_i(t+1)H_i({}^ie^-(t+1) + v_i(t+1)) \end{aligned} \qquad (14) $$

Like the Kalman gain, $L_i(t+1)$ can be computed in a way minimizing the mean-squared error of the state estimate, i.e., $\mathbb{E}[\|{}^ie(t+1)\|^2]$. This is in fact equivalent to minimizing the trace of the posterior covariance matrices, i.e., $\mathrm{Tr}({}^i\Sigma(t+1)), \forall i \in \mathcal{V}$. By the definition of ${}^i\Sigma(t+1)$, we have:

$$ \begin{aligned} {}^i\Sigma(t+1) &:= \mathbb{E}[{}^ie(t+1){}^ie(t+1)^{\mathrm{T}} | Z_{i,(0:t+1)}] \\ &= (I_{nN} - L_i(t+1)H_i){}^i\Sigma^-(t+1)(I_{nN} - L_i(t+1)H_i)^{\mathrm{T}} \\ &\quad + L_i(t+1)H_i{}^i\Xi(t+1)(L_i(t+1)H_i)^{\mathrm{T}} \end{aligned} \qquad (15) $$

where

$$ \begin{aligned} L_i(k+1) &= {}^i\Sigma^-(t+1)H_i^{\mathrm{T}}(S_i(t+1))^{-1} \\ S_i(t+1) &= H_i({}^i\Sigma^-(t+1) + {}^i\Xi(x+1))H_i^{\mathrm{T}} \\ {}^i\Xi(t+1) &= blkdg(\Xi_{i1}(t+1),\ \Xi_{i2}(t+1), \dots \Xi_{iN}(t+1)) \end{aligned} \qquad (16) $$

Correspondingly, $\Sigma(t+1)$ can be updated by:

$$ \begin{aligned} \Sigma(t+1) &= (I - \tilde{L}(t+1))\Sigma^-(t+1)(I - \tilde{L}(t+1))^{\mathrm{T}} \\ &\quad + \tilde{L}(t+1)\Sigma_{\Xi}(t+1)\tilde{L}(t+1)^{\mathrm{T}} \end{aligned} \qquad (17) $$

where $\Sigma_{\Xi}(t+1) = blkdg({}^1\Xi(t+1), \dots, {}^N\Xi(t+1))$, and $\tilde{L}(t+1) = blkdg(L_1(t+1)H_1, \dots, L_N(t+1)H_N)$. Note that, the covariance between the state estimation errors at current and past steps, i.e., $\mathbb{E}[e(t{+}1)e(s)^{\mathrm{T}}]$ and $\mathbb{E}[e(s)e(t{+}1)^{\mathrm{T}}], \forall s < t$ need to be updated using the computed $\tilde{L}(t+1)$. The cross-covariance between the i^{th} and the j^{th} agents' state estimates ${}^{ij}\Sigma(t+1) := \mathbb{E}[{}^ie(t+1){}^je(t+1)^{\mathrm{T}}] \in \mathbb{R}^{Nn\times Nn}$ is at the off-diagonal block entry, while ${}^i\Sigma(t+1) \in \mathbb{R}^{Nn\times Nn}$ is at the diagonal block entry of $\Sigma(t+1) \in \mathbb{R}^{NnN\times NnN}$. Due to space limit, the detailed expansions of them are reported in the Arxiv version [11]. Based on the computed Υ_i from (16), each agent can update ${}^i\hat{x}(t)$, ${}^i\Sigma(t)$ and $\Sigma(t)$, respectively using (14), (15), and (17). This completes the implementation of the distributed estimation algorithm.

Remark 2: It is noted that $\Sigma(t)$ computed by each agent is irrespective of agent's perspective since the same initial condition $\Sigma(0)$ is given to each agent by Assumption 1.

Remark 3: The proposed distributed estimator guarantees the stability of the estimation error in the sense of Lyapunov, if individual agents satisfy the observability condition stated in [9, Lemma 1]. The full proof of the stability is reported in the Arxiv version [11].

B. Distributed Control Law Design

In this section, the computationally tractable formulation of the optimal distributed control law is derived from the individual agents' perspectives. The main idea starts with relaxing the structural constraints on $\mathcal{F}$ by applying the estimator (14) to each agent.

Definition 3: Let ${}^ie := x - {}^i\hat{x}$ stacks up the time series of the estimation errors from the i^{th} agent's perspective, over the time horizon T. Given $\mathcal{F}$ and $\Upsilon_i, \forall i \in \mathcal{V}$, one can construct the estimation error covariance over the time horizon T, $\Sigma_i := \mathbb{E}[{}^ie{}^ie^{\mathrm{T}}], \forall i \in \mathcal{V}$, as well as the cross-covariance between two different agents $\Sigma_{ij} := \mathbb{E}[{}^ie\ {}^je^{\mathrm{T}}], \forall i \neq j \in \mathcal{V}$.

In terms of the time series of the estimation errors, the state estimation based control law over the time horizon can be expressed by:

$$ u = \sum_i^N \mathcal{M}_i\mathcal{F}C\ {}^i\hat{x} = \mathcal{F}Cx - \sum_i^N \mathcal{M}_i\mathcal{F}C\ {}^ie \qquad (18) $$

where $\mathcal{M}_i = I_T \otimes (M_i^{\mathrm{T}}M_i), \forall i \in \mathcal{V}$. Plugging (18) into (4) yields the objective cost in (8) as follows:

$$ \begin{aligned} J(\mathcal{F}, \Upsilon_1, \dots, \Upsilon_N) &= \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1}P_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}CP_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 + \sum_{i,j}\|\mathcal{Q}^{\frac{1}{2}}P_{12} \\ &\times\ (I - \mathcal{F}CP_{12})^{-1}(\mathcal{M}_i\mathcal{F}C\Sigma_{ij}C^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{M}_j^{\mathrm{T}})^{\frac{1}{2}}\|_F^2 \\ &+ \sum_{i,j}\|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}(\mathcal{M}_i\mathcal{F}C\Sigma_{ij}C^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{M}_j^{\mathrm{T}})^{\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1}P_{11}\mu_w\|_2^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}CP_{11}\mu_w\|_2^2 \end{aligned} \qquad (19) $$

where $\|\cdot\|_2^2$ and $\|\cdot\|_F^2$ denote Euclidean norm and the Frobenius norm, respectively; and $\mu_w := \mathbb{E}[w] \in \mathbb{R}^{Nn(T+1)}$, $\Sigma_w := \mathbb{E}[(w-\mu_w)(w-\mu_w)^{\mathrm{T}}] \in \mathbb{R}^{Nn(T+1)\times Nn(T+1)}$.

Apparently, the objective cost (19) has high-dimensional, highly coupled optimization variables, i.e., $\mathcal{F}$, which is our main interest, and $\Sigma_{ij}, \forall i,j \in \mathcal{V}$, which are the implicit functions of both $\mathcal{F}$ and $\Upsilon_i, \forall i \in \mathcal{V}$. The proposed iterative optimization procedure alleviates these coupling complexities in two aspects. First, akin to the alternating direction method of multipliers (ADMM) technique [12], we set $\Sigma_{ij}, \forall i,j \in \mathcal{V}$ constant while optimizing $\mathcal{F}$ at the l^{th} iteration, thereby treating J as the function of $\mathcal{F}$ only. Note that $\Upsilon_i, \forall i \in \mathcal{V}$ is designed over the constant $\mathcal{F}$ in the *distributed estimator design* phase of the next iteration. Second, we interpret the global objective cost from the individual agent's viewpoint, and translate the primal problem (Problem 2) into the agent-wise objective cost. The objective cost, locally seen by the i^{th} agent's viewpoint at the l^{th} iteration, is denoted by ${}^iJ^{(l)}$ which can be constructed using the estimated MAS input ${}^iu^{(l)} = {}^i\mathcal{F}^{(l)}C(x - {}^ie)$ instead of (18). ${}^i\mathcal{F}^{(l)}$ is constructed from the i^{th} agent's perspective by optimizing the agent-wise objective cost, ${}^iJ^{(l)}$. Then, the resulting agent-wise optimization problem is represented as follows:

Problem 3 (Optimal Distributed Control Law From Agent-Wise Viewpoint):

$$\min_{{}^i\mathcal{F}^{(l)}\in\tilde{\mathbb{F}}} \ {}^iJ^{(l)}({}^i\mathcal{F}^{(l)}) \tag{20}$$

where

$$\begin{aligned}{}^iJ^{(l)}({}^i\mathcal{F}^{(l)}) &= \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\,{}^i\mathcal{F}^{(l)}C)^{-1}P_{11}\,{}^i\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - {}^i\mathcal{F}^{(l)}CP_{12})^{-1}\,{}^i\mathcal{F}^{(l)}CP_{11}\,{}^i\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\,{}^i\mathcal{F}^{(l)}C)^{-1}P_{12}\,{}^i\mathcal{F}^{(l)}C\Sigma_i^{(l)\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - {}^i\mathcal{F}^{(l)}CP_{12})^{-1}\,{}^i\mathcal{F}^{(l)}C\Sigma_i^{(l)\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\,{}^i\mathcal{F}^{(l)}C)^{-1}P_{11}\,{}^i\mu_w\|_2^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - {}^i\mathcal{F}^{(l)}CP_{12})^{-1}\,{}^i\mathcal{F}^{(l)}CP_{11}\,{}^i\mu_w\|_2^2 \end{aligned} \tag{21}$$

where ${}^i\mu_w = \mathbb{E}[w|Z_{i,(0:T)}]$, ${}^i\Sigma_w := \mathbb{E}[(w - {}^i\mu_w)(w - {}^i\mu_w)^{\mathrm{T}}|Z_{i,(0:T)}], \forall i \in \mathcal{V}$. Note that $\Sigma_i^{(l)} \in \mathbb{R}^{Nn(T+1)\times Nn(T+1)}$ is computed by Definition 3 at the l^{th} iteration.

Definition 4: A subspace $\tilde{\mathbb{F}}$ is quadratic invariance (QI) with respect to CP_{12} if and only if ${}^i\mathcal{F}^{(l)}CP_{12}{}^i\mathcal{F}^{(l)} \in \tilde{\mathbb{F}}$. And it is trivial to show that $\tilde{\mathbb{F}}$ is QI with respect to CP_{12}[13].

It is well-known fact that QI is a sufficient and necessary condition for the exact convex reformulation [13]. That is, one can apply an equivalent disturbance-feedback policy to make (21) a convex form, similar to [10].

Definition 5: Let us introduce the nonlinear mapping as:

$$h(\Phi) = (I + \Phi CP_{12})^{-1}\Phi, \ h : \mathbb{R}^{NpT\times NnT} \mapsto \mathbb{R}^{NpT\times NnT} \tag{22}$$

and define the cost function $\tilde{J} : \mathbb{R}^{NpT\times NnT} \mapsto \mathbb{R}$ in terms of the design parameter ${}^i\Phi^{(l)}$[10].

$$\begin{aligned}{}^i\tilde{J}^{(l)}({}^i\Phi^{(l)}) &= \|\mathcal{Q}^{\frac{1}{2}}(I + P_{12}{}^i\Phi^{(l)}C)P_{11}{}^i\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}{}^i\Phi^{(l)}CP_{11}{}^i\Sigma_w^{\frac{1}{2}}\|_F^2 + \|\mathcal{Q}^{\frac{1}{2}}P_{12}{}^i\Phi^{(l)}\Sigma_i^{(l)\frac{1}{2}}\|_F^2 \\ &+ \|\mathcal{R}^{\frac{1}{2}}{}^i\Phi^{(l)}\Sigma_i^{(l)\frac{1}{2}}\|_F^2 + \|\mathcal{R}^{\frac{1}{2}}{}^i\Phi^{(l)}CP_{11}{}^i\mu_w\|_2^2 \\ &+ \|\mathcal{Q}^{\frac{1}{2}}(I + P_{12}{}^i\Phi^{(l)}C)P_{11}{}^i\mu_w\|_2^2 \end{aligned} \tag{23}$$

By [10, Th. 1], the following convex optimization problem is equivalent to Problem 3.

Problem 4 (Equivalent Convex Problem to Optimal Distributed Control From Agent-Wise Viewpoint):

$$\min_{{}^i\Phi^{(l)}\in h^{-1}(\tilde{\mathbb{F}})} \ {}^i\tilde{J}^{(l)}({}^i\Phi^{(l)}) \tag{24}$$

By solving (24) using convex programming, one can find the optimal ${}^i\Phi^{(l)}$, and the corresponding ${}^i\mathcal{F}^{(l)}$ from the inverse mapping h^{-1} of (22). The same optimization routines (Problem 3 and 4) based on the locally seen cost from the other agents' perspectives are processed to get the optimal control laws ${}^i\mathcal{F}^{(l)}, \forall i \in \mathcal{V}$ at the l^{th} iteration step.

C. Distributed Control-Estimation Synthesis

The set of optimal control laws from individual agents' viewpoint, ${}^i\mathcal{F}^{(l)}, \forall i \in \mathcal{V}$, is mixed to approximate the solution to Problem 2 by the agent-wise mixing strategy proposed as follows:

$$\mathcal{F}^{(l)} = \sum_i^N \mathcal{M}_i\,{}^i\mathcal{F}^{(l)} \tag{25}$$

The basic intuition of the proposed strategy is to exhibit the control law for the i^{th} agent using the one computed from the sup-optimization problem (Problem 3) from the i^{th} agent's perspective. Accordingly, the proposed mixing strategy allows for individual agents to retain distributed controllers to be executed, retaining each of their sub-optimal solutions without interfering with each other.

D. Convergence Check

The last step of the iteration loop evaluates the designed distributed control law (25), together with the estimator (7). First, let S be a set which stores the designed control law, $\mathcal{F}^{(l)}$, and the estimator gains, $\Upsilon_i^{(l)}, \ \forall i \in \mathcal{V}$, from each iteration step as follows:

$$S := \left\{ s^{(l)} \middle| s^{(l)} = \left(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \dots, \Upsilon_N^{(l)}\right), \ l \in \mathbb{N} \right\} \tag{26}$$

The iteration terminates if: i) the total iteration counts the threshold number N_{max}; or ii) the consecutive iteration is converged with respect to the following stopping condition:

$$\triangle J(l, l-1) \le \epsilon_{stop} \tag{27}$$

where $\triangle J(l, l-1) := |J(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \dots, \Upsilon_N^{(l)}) - J(\mathcal{F}^{(l-1)}, \Upsilon_1^{(l-1)}, \dots, \Upsilon_N^{(l-1)})|$ and ϵ_{stop} is the threshold magnitude for the convergence. The objective cost of the corresponding control law $J(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \dots, \Upsilon_N^{(l)})$ is computed by plugging the designed control law, $\mathcal{F}^{(l)}$, and the set of distributed estimator gains $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$ into (19). The final

Algorithm 1: Virtual Interaction Based Distributed Control-Estimation Synthesis

Initialize the MAS dynamics information A, B, $\mathcal{L}$, $\Sigma(0)$, $\mathcal{F}^{(0)}$, ϵ_{stop}, N_{max} and the cost metrics $\mathcal{Q}$, $\mathcal{R}$.
for $l = 1, 2, \cdots N_{max}$ do
a) Distributed estimator design
for t=0 to the termination time T
1) Update $\Sigma(t+1)$ using $\mathcal{F}^{(l-1)}$, (13), (16), and (17)
end for, Output $\longrightarrow \Upsilon_i^{(l)}$ and $\Sigma_i^{(l)}, \forall i \in \mathcal{V}$
b) Distributed control law design
for i=1 to the number of total agent, N
2) Solve (24), and compute ${}^i\mathcal{F}^{(l)}$
end for, Output $\longrightarrow {}^i\mathcal{F}^{(l)}, \forall i \in \mathcal{V}$
c) Distributed control-estimation synthesis
3) Synthesize the control law $\mathcal{F}^{(l)}$ using (25)
d) Convergence check
4) Store $\mathcal{F}^{(l)}$, and $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$ in the set S
5) If (27) is satisfied or $l > N_{max}$, then terminates.
end for, **Output** $\Longrightarrow \mathcal{F}^*$, and Υ_i^*, $\forall i \in \mathcal{V}$

output of the control-estimation synthesis is given by:

$$\mathcal{F}^* = \mathcal{F}^{(l)}, \;\; \Upsilon_i^* = \Upsilon_i^{(l)}, \;\; \forall i \in \mathcal{V}$$
$$\text{where } l = \underset{\forall l \in |S|}{\arg\min}\, J(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \dots, \Upsilon_N^{(l)}) \tag{28}$$

The overall recursive structure of the control-estimation synthesis procedure is summarized in Algorithm 1.

It is noted that the Algorithm 1 is executed in the offline design phase. Once the distributed control law $\mathcal{F}^*$ and the corresponding estimator gains Υ_i^* for each agent are designed, each agent is deployed into the distributed online operation using its own prior knowledge. It is worth noting that the majority of the heavy computations occur at the offline design phase. Therefore, when it comes to the online operation, it is not burdensome to the limited on-board resources of each agent. Indeed, the computational complexity of the online operation for the proposed algorithm is scaled by the number of agents, i.e., $\mathcal{O}(N)$.

IV. Numerical Simulation

The effectiveness of the proposed algorithm is demonstrated with an illustrative MAS example. The MAS consists of five agents whose dynamics and objective are specified by the following parameter sets: $A = 1$, $B = 1$, $\Theta_i(t) = 1$, ${}^i\Xi(t) = diag(1, 1, 1, 2, 1), \forall i \in \mathcal{V}$, $\forall t \in \{0, \dots, T\}$, $T = 5$, $\mathcal{Q} = I_6 \otimes (5I_5 - 1_{5\times 5})$, and $\mathcal{R} = I_{25}$. The MAS network topology is set to be partially connected, the same as the one in [9]. To validate the performance of the proposed algorithm, we conduct a comparative analysis with two different scenarios: i) MAS with the fully connected network, which is free from network topological constraint; and ii) MAS with the same (partially connected) network topology where non-neighboring agent information is not available to each agent. For the second case, we test the suboptimal method presented in [5]. The simulation results are shown in Fig. 3. By virtue of the virtual interactions between non-neighboring agents, our algorithm outperforms the existing method in the partially connected network, and even matches the fully connected network case despite the network topological constraints.

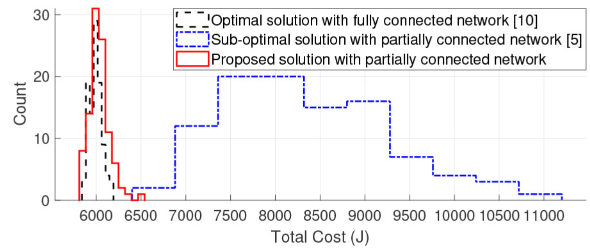

Fig. 3. Cost value statistics histogram (Monte Carlo simulations with 100 runs).

V. Conclusion

This letter has proposed a novel design procedure of the optimal distributed control for the linear stochastic MAS, generally subject to network topological constraints. The proposed method gets around the network topological constraint by employing the distributed estimator, whereby each agent can exploit the non-neighboring agent's information. Future work will include the theoretical performance guarantee of the proposed distributed control-estimation synthesis such as stability analysis, and a further extension to the infinite time horizon case for practical use.

References

[1] P. Shi and B. Yan, "A survey on intelligent control for multiagent systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 1, pp. 161–175, Jan. 2021.
[2] X. Li, F. Liu, M. Buss, and S. Hirche, "Fully distributed consensus control for linear multi-agent systems: A reduced-order adaptive feedback approach," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 2, pp. 967–976, Jun. 2020.
[3] Y. Zhu, L. Zhou, Y. Zheng, J. Liu, and S. Chen, "Sampled-data based resilient consensus of heterogeneous multiagent systems," *Int. J. Robust Nonlinear Control*, vol. 30, no. 17, pp. 7370–7381, 2020.
[4] R. Morita, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki, "Multiagent consensus with noisy communication: Stopping rules based on network graphs," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 4, pp. 358–365, Dec. 2016.
[5] V. Gupta, B. Hassibi, and R. M. Murray, "A sub-optimal algorithm to synthesize control laws for a network of dynamic agents," *Int. J. Control*, vol. 78, no. 16, pp. 1302–1313, 2005.
[6] J. Ma, Y. Zheng, and L. Wang, "LQR-based optimal topology of leader-following consensus," *Int. J. Robust Nonlinear Control*, vol. 25, no. 17, pp. 3404–3421, 2015.
[7] D. H. Nguyen, "A sub-optimal consensus design for multi-agent systems based on hierarchical LQR," *Automatica*, vol. 55, pp. 88–94, May 2015.
[8] J. Jiao, H. L. Trentelman, and M. K. Camlibel, "A suboptimality approach to distributed linear quadratic optimal control," *IEEE Trans. Autom. Control*, vol. 65, no. 3, pp. 1218–1225, Mar. 2020.
[9] C. Kwon and I. Hwang, "Sensing-based distributed state estimation for cooperative multiagent systems," *IEEE Trans. Autom. Control*, vol. 64, no. 6, pp. 2368–2382, Jun. 2019.
[10] L. Furieri and M. Kamgarpour, "First order methods for globally optimal distributed controllers beyond quadratic invariance," in *Proc. IEEE Amer. Control Conf. (ACC)*, 2020, pp. 4588–4593.
[11] H. Lee and C. Kwon, "Distributed control-estimation synthesis for stochastic multi-agent systems via virtual interaction between non-neighboring agents," 2021. [Online]. Available: arXiv:2106.00961.
[12] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2426–2431, Sep. 2013.
[13] L. Lessard and S. Lall, "Quadratic invariance is necessary and sufficient for convexity," in *Proc. IEEE Amer. Control Conf.*, 2011, pp. 5360–5362.